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Weakly Commuting Mappings in Digital Metric Spaces

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Abstract: In this paper, we prove a common fixed point theorem for weakly commuting mappings in digital metric spaces and give an example in support of our result.

Keywords: Fixed point, Digital topology, Digital contraction, weakly Commuting mappings.

2010 Mathematical Subject Classification: 47H10, 54E35, 68U10.

1. INTRODUCTION

Topology is the study of geometric properties that does not depend only on the exact shape of the objects, but rather it acts on how the points are connected to each other. Infact, topology deals with those properties that remain invariant under the continuous transformation of a map. In 1979, Rosenfeld [12] introduced the concept of Digital Topology. Digital topology is concerned with geometric and topological properties of digital image. The digital images have been used in computer sciences (image processing and computer graphics). Digital topology also provides a mathematical basis for image processing operation in 2D and 3D digital images. For more detail, one can refer to [1, 7, 11].

In topology, infinitely many points are considered in arbitrary small neighborhood of a point but digital topology is concerned with finite number of points in a neighbourhood of a point. Therefore, one can distinguish easily between general topology and digital topology by considering the neighbourhood of a point as shown in the folowing figures.

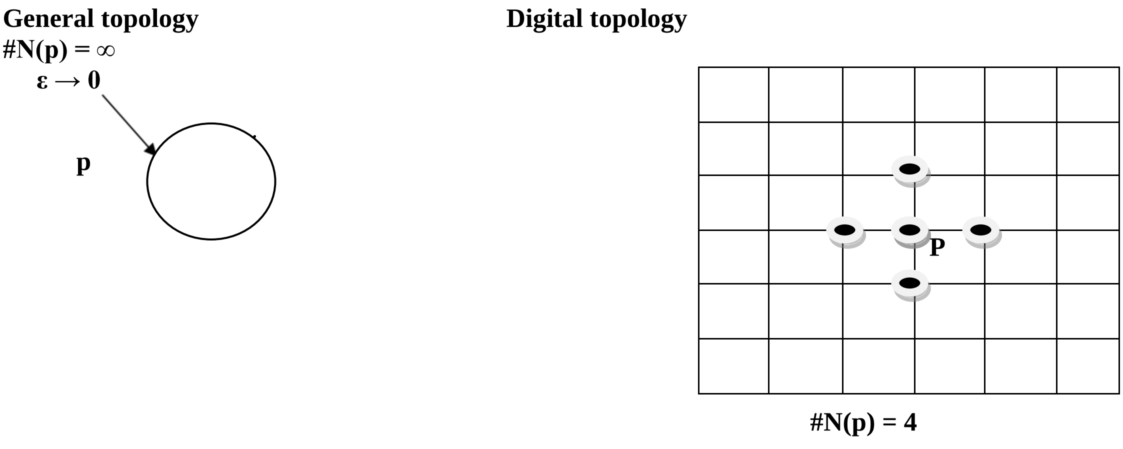

FIGURE 1. Neighboorhood in General and Digital topology

Digital image processing is a rapidly growing discipline with many applications in business (document reading), industry (automated assembly and inspection), medicine (radiology, haematology, etc.), and the environmental sciences (metrology, geology, land use management, etc.) and among many other fields. The work involves the analysis of picture i.e., the regions of which it is composed. A picture is input to the computer by sampling its brightness values at a discrete grid of points and digitizing or quantizing these values into binary digits. The result of this process is called a digital picture; it is a rectangular array of discrete values. The elements of this array are called pixels and the value of a pixel is called its gray level. The process of decomposing a picture into regions is called segmentation. Segmentation is basically a process of assigning the pixels. The one simple way of doing this process is called thresholding.

Once a picture has been segmented into subsets then it can be described by properties of subsets. Some of these properties depend on the gray levels of the points and some on the positions of the points. Basically, digital topology involves the concept of adjacency (surrounding) but not size or shape. The adjacency relations among the regions can be compactly represented by a graph. The two nodes of a graph are joined by an arc if those two regions are adjacent.

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2. TOPOLOGICAL VIEW POINT OF DIGITAL METRIC SPACES

Let $\mathbb{Z}^n$, $n \in \mathbb{N}$, be the set of points in the Euclidean n dimensional space with integer coordinates.

Definition 2.1. [4] Let l, n be positive integers with $1 \leq l \leq$ n. Consider two distinct points

$$p = (p_1, p_2, \dots p_n),\ q = (q_1, q_2, \dots q_n) \in \mathbb{Z}^n$$

The points p and q are k_l-adjacent if there are at most l indices i such that $|p_i - q_i| = 1$, and for all other indices j,$|p_j - q_j| \neq 1$, $p_j = q_j$.

(i) Two points p and q in $\mathbb{Z}$ are 2-adjacent if |p – q|=1 (see Figure 2).

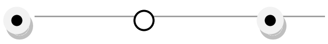

FIGURE 2. 2-adjacency

(ii) Two points p and q in $\mathbb{Z}^2$ are 8-adjacent if the points are distinct and differ by at most 1 in each coordinate i.e., the 4-neighbors of (x, y) are its four horizontal and vertical neighbors $(x \pm 1, y)$ and$(x, y \pm 1)$.
(iii) Two points p and q in $\mathbb{Z}^2$ are 4-adjacent if the points are 8-adjacent and differ in exactly one coordinate i.e., the 8-neighbors of (x, y) consist of its 4-neighbors together with its four diagonal neighbors $(x + 1, y \pm 1)$ and$(x - 1, y \pm 1)$. (see Figure 3).

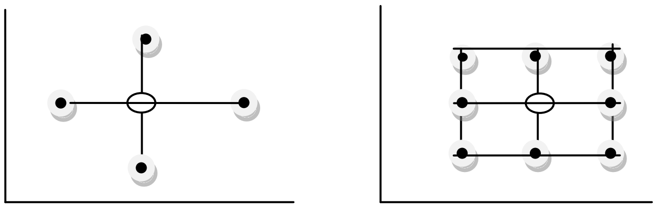

FIGURE 3. 4-adjacency and 8-adjacency

(iv) Two points p and q in $\mathbb{Z}^3$ are 26-adjacent if the points are distinct and differ by at most 1 in each coordinate. i.e.,
(a) Six face neighbours $(x \pm 1, y, z)$, $(x, y \pm 1, z)$ and $(x, y, z \pm 1)$
(b) Twelve edge neighbours $(x \pm 1, y \pm 1, z)$, $(x, y \pm 1, z \pm 1)$
(c) Eight corner neighbours $(x \pm 1, y \pm 1, z \pm 1)$
(v) Two points p and q in $\mathbb{Z}^3$ are 18-adjacent if the points are 26-adjacent and differ by at most 2 coordinate. i.e.,
(a)Twelve edge neighbours $(x \pm 1, y \pm 1, z)$, $(x, y \pm 1, z \pm 1)$
(b) Eight corner neighbours $(x \pm 1, y \pm 1, z \pm 1)$
(vi) Two points p and q in $\mathbb{Z}^3$ are 6-adjacent if the points are 18-adjacent and differ in exactly one coordinate. i.e.,
(a) Six face neighbours $(x \pm 1, y, z)$, $(x, y \pm 1, z)$ and $(x, y, z \pm 1)$ (See Figure 4).

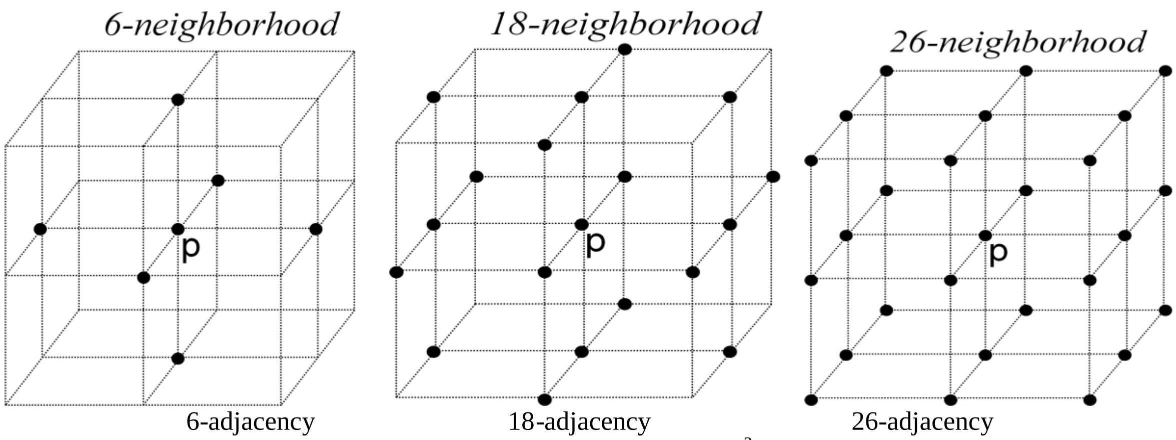

FIGURE. 4. Adjacencies in $\mathbb{Z}^3$

One can easily note that the coordination number of Na in the crystal structure of NaCl is 6 which is equal to adjacency relation in digital images of figure 5.

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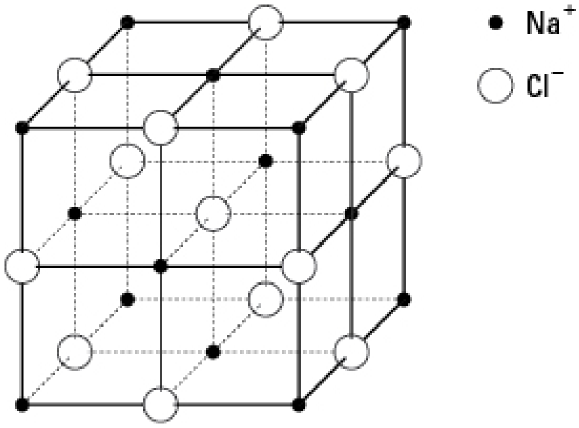

FIGURE 5. Crystal structure of NaCI

Definition 2.2. Let $\emptyset \neq X \subset \mathbb{Z}^n$, $n \in N$. A digital image is a pair (X, k), where k is an adjacency relation on X. Technically, a digital image (X, k) is an undirected graph whose vertex set is the set of members of X and whose edge set is the set of unordered pairs $\{x_{0,}x_1\} \subset X$ such that $x_0 \neq x_1$ and x_0 and x_1 are $k-$ adjacent.

The notion of digital continuity in digital topology was developed by Rosenfeld [13] to study 2D and 3D digital images. Boxer [2] gives the digital version of several notions of topology and Ege and Karaca [5] described the digital continuous functions.

Let $\mathbb{N}$ and $\mathbb{R}$ denote the sets of natural numbers and real numbers, respectively. Boxer [3] defined a k – neighbor of a point $p \in \mathbb{Z}^n$.

A k – neighbor of a point $p \in \mathbb{Z}^n$ is a point of $\mathbb{Z}^n$ that is k - adjacent to p, where $k \in \{2,4,6\ 8,18,26\}$ and $n \in \{1, 2, 3\}$.

The set $N_k\,(p) = \{q \mid q \text{ is } k \text{ - adjacent to } p\}$ is called the k -neighborhood of p.

Boxer [2] defined a digital interval as

$$[a, b]_{\mathbb{Z}} = \{z \in \mathbb{Z} \mid a \leq z \leq b\}, \text{where } a, b \in \mathbb{Z} \text{ and } a < b.$$

A digital image $X \subset \mathbb{Z}^n$ is k-connected [8] if and only if for every pair of distinct points $x, y \in X$, there is a set $\{x_{0,}x_{1,}x_{2\,,\ldots,}x_r\}$ of points of a digital image X such that $x = x_0, y = x_r$ where x_i and x_{i+1} are k -neighbors and $i =$ 0, 1... r- 1.

Definition 2.3. Let $(X,k_0) \subset \mathbb{Z}^{n_0}$, $(Y, k_1) \subset \mathbb{Z}^{n_1}$ be digital images and f: $X \rightarrow Y$ be a function.

(i)If for every k_0-connected subset U of X, $f\ (U)$ is a k_1-connected subset of Y, then f is said to be (k_0,k_1)-continuous [3].

(ii) f is (k_0,k_1)-continuous for every k_0-adjacent points $\{x_{0,}x_1\}$ of X, either $f(x_0) = f(x_1)$ or $f(x_0)$ and $f(x_1)$ are k_1-adjacent in Y[3] .

(iii) If f is (k_0,k_1)-continuous, bijective and f^{-1} is (k_0,k_1)-continuous, then f is called (k_0,k_1)-isomorphism and denoted by $\cong_{(k_0,k_1)} Y$.

Now we start with digital metric space(X, d, k) where d is usual Euclidean metric on $\mathbb{Z}^n$ and k denote the adjacency relation among the points in $\mathbb{Z}^n$.

Definition 2.4. [5] Let (X, k) be a digital images set. Let d be a function from $(X, k) \times (X, k) \rightarrow \mathbb{Z}^n$ satisfying all the properties of a metric space. The triplet (X, d, k) is called a digital metric space.

Proposition 2.5. [7] Let (X, d, k) be a digital metric space. A sequence $\{x_n\}$ of points of a digital metric space (X, d, k) is

(i) a Cauchy sequence if and only if there is $\alpha \in \mathbb{N}$ such that for all, $\gneqq \alpha$, then

$$d(x_n\,,x_m) \lneqq 1 \text{ i.e., } x_n = x_m.$$

(ii) convergent to a point $l \in X$ if for all $\epsilon \gneqq 0$, there is $\alpha \in \mathbb{N}$ such that for all $n \gneqq \alpha$ then $d(x_n\,,l) \lneqq \epsilon\,, i.e. x_n = l\,.$

Proposition 2.6. [7] A sequence $\{x_n\}$ of points of a digital metric space (X, d, k) converges to a limit $l \in X$ if there is $\alpha \in \mathbb{N}$ such that for all $\gneqq \alpha$, then $x_n = l.$

Theorem 2.7. [7] A digital metric space (X, d, k) is complete.

Definition 2.8. [5] Let (X, d, k) be any digital metric space. A self map f on a digital metric space is said to be digital contraction, if there exists a $\lambda \in [0, 1)$ such that for all $x, y \in X$,

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$$d(f(x), f(y)) \leq \lambda\, d(x, y)$$

Proposition 2.9. [5] Every digital contraction map $f: (X, d, k) \rightarrow (X, d, k)$ is digitally continuous.

Proposition 2.10. [7] In a digital metric space (X, d, k), consider two points x_i, x_j in a sequence $\{x_n\} \subset X$ such that they are k -adjacent. Then they have the Euclidean distance $d(x_i, x_j)$ which is greater than or equal to 1 and at most √ t depending on the position of the two points.

Definition 2.11. In 1982 (Sessa [14]) Two self mappings f and g of a digital metric space (X, d, k) are called weakly commuting iff $d(fgx, gfx) \leq d(fx, gx)$ for all x in X

3. WEAKLY COMMUTING MAPPINGS AND COMMON FIXED POINTS

In 1976 Jungck [9] obtained the common fixed point for commuting mappings by using a constructive procedure of sequence of iterates.

The first ever attempt to relax the commutativity of mappings to a smaller subset of domain of mappings was initiated by Sessa [14] who in 1982 give the notion of weak commutativity as given the definition 2.11 above.

Now, we focus ourselves to prove fixed point theorem for weakly commuting mappings in setting of digital metric spaces as follows:

Theorem 3(A). Let $\emptyset \neq X \subset \mathbb{Z}^n$, $n \in N$ and (X, k) be a digital image and k is an adjacency relation in X. Let S, T be mappings of a complete digital metric space (X, d, k) into itself satisfying the following conditions:

(3.1) $T(X) \subseteq S(X)$;

(3.2) S is (k, k)continuous;

(3.3) there exist $0 < \alpha < 1$ such that, for all $x, y \in X$,

$$d\big(T(x), T(y)\big) \leq \alpha d\big(S(x), S(y)\big).$$

Then S and T have a unique common fixed point in X provided S and T weakly commute on X.

Proof. Let $x_0 \in X$. By (3.1) we can find x_1 such that $S(x_1) = T(x_0)$. For this x_1 we can find $x_2 \in X$ such that $S(x_2) = T(x_1)$. In general, choose $\{x_{n+1}\}$ in X such that $S(x_{n+1}) = T(x_n), n = 0,1,2 \ldots$

Consider,

$$\begin{aligned} d(S(x_n), S(x_{n+1})) &= d\big(T(x_{n-1}), T(x_n)\big) \\ &\leq \alpha d(S(x_{n-1}), S(x_n)) \\ &\leq \cdots \\ &\leq \alpha^n d\big(S(x_0), S(x_1)\big). \end{aligned}$$

Hence for any positive integer p,

$$\begin{aligned} d\left(S(x_n), S(x_{n+p})\right) &\leq d\big(S(x_n), S(x_{n+1})\big) + d\big(S(x_{n+1}), S(x_{n+2})\big) + \ldots d\left(S(x_{n+p-1}), S(x_{n+p})\right) \\ &\leq (\alpha^n + \alpha^{n+1} + \alpha^{n+2} + \cdots . + \alpha^{n+p-1}) d(S(x_0), S(x_1)) \\ &\leq \frac{\alpha^n}{1-\alpha}\, d(S(x_0), S(x_1)). \end{aligned}$$

Since $0 < \alpha < 1$, therefore, as n→∞, we have $d\left(S(x_n), S(x_{n+p})\right) \rightarrow 0$

Thus $\{S(x_n)\}$ is a Cauchy sequence in (X, d, k) and due to completeness property of digital metric space, (X, d, k), $\{S(x_n)\}$ converges to a point z and $T(x_n) = S(x_{n+1})$ also converges to the same point z. From (3.2) the (k, k) continuity of S implies the (k, k) continuity of T. Therefore, $\{T(S(x_n))\}$ converges to $T(z)$. However, since S and T weakly commute on X, therefore,

$$d\left(S\big(T(x_n)\big), T\big(S(x_n)\big)\right) \leq d\left(\big(S(x_n)\big), T\big((x_n)\big)\right)$$

$$\Longrightarrow\ d(T(z), S(z)) \leq d(z, z)$$

$$\Longrightarrow\quad T(z) = S(z)$$

So z is a coincidence point of S and T. So, $S\big(T(z)\big) = T\big(S(z)\big) = T\big(T(z)\big)$. We can therefore infer

$$d(T(z), T(T(z)) \leq \alpha d\big(S(z), S(T(z)\big) = \alpha d\big(T(z), T(T(z)\big).$$

Hence $\qquad T(z) = T\big(T(z)\big) = S(T(z)).$

Therefore, T(z) is a common fixed point of S and T

Uniqueness:

Suppose $x \neq z$ be two fixed point of S and T, therefore, $x = S(x) = T(x)$ and $z = T(z) = S(z)$. From (3.3), we have $d(x, z) = d\big(T(x), T(z)\big) \leq \alpha d\big(S(x), S(z)\big) = \alpha d(x, z)$, i. e., $x = z$. This completes the proof.

Example3.4. Consider the minimal simple closed 18-surface $MSS'_{18} = \{c_i : i \in [0,5]_{\mathbb{Z}}\}$ (see Figure 6).

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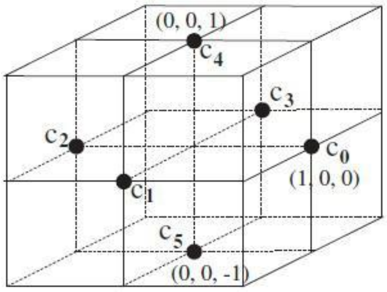

FIGURE 6. MSS'_{18} [6]

Let S: $MSS'_{18} \to MSS'_{18}$ and T: $MSS'_{18} \to MSS'_{18}$ be digital map satisfying the inequality (3.3).
Consider a point such as c_0 in MSS'_{18} and take $S(c_0) = c' \in MSS'_{18}$ and $T(c_0) = c'' \in MSS'_{18}$. For the point $c_i \in N_{18}(c_0, 1), i \in \{1,3,4,5\}$, we have

$$d(T(c_i), T(c_0)) \leq \alpha\, d(S(c_i), S(c_0))$$
$$\leq \alpha d(S(c_i), c')$$
$$\leq \alpha \sqrt{2}, \text{ from Proposition 2.11}$$

Since $0 < \alpha < \frac{1}{2}$, we get $d\big(T(c_i), T(c_0)\big) \lneqq \sqrt{2}$. As a result, $d\big(T(c_i), T(c_0)\big) = 0$ implies that $T(c_i) = T(c_0) = c'$ from the property of MSS'_{18}. This procedure can be applied to all point in MSS'_{18} since, c_0 is an arbitrary point. Therefore, S is also constant maps. By the Theorem 3(A) we can say that S and T has a unique common fixed point.

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